Ensuring Network Connectivity During Formation Control Using A Decentralized Navigation Function

Z. Kan, A. P. Dani, J. M. Shea, and W. E. Dixon

Abstract— **In many applications of formation control, agents coordinate and communicate to make appropriate decisions. Connectivity of the network is paramount in such applications. The goal in this paper is to drive a group of agents with limited sensing capabilities to a desired configuration while ensuring the connectivity of the wireless communication network among the agents. Based on a navigation function formalism, a decentralized cooperative controller is proposed where agent only uses information within its sensing zone to guarantee connectivity maintenance of the network and achieve the desired formation with collision avoidance between themselves and with obstacles in the environment.**

I. Introduction

In multi-agent cooperative control, agents coordinate and communicate to achieve a collective goal (e.g., flocking, consensus, or pattern formation). As agents move to perform some mission objective, ensuring the group remains close enough to maintain radio communication (i.e., the group does not partition) can be challenging in a decentralized control system.

The use of an artificial potential field is one method that has been widely used for formation control. The potential function produces a repulsive potential field around a workspace boundary and obstacles, and an attractive potential field is produced at the goal configuration. A common problem with artificial potential field-based formation control algorithms is the existence of local minima when attractive and repulsive force are combined [1]. In the seminal work in [2] and [3], a navigation function approach is developed for a single point-mass agent moving in an environment with spherical obstacles. The navigation function proposed is a real valued function which is designed such that the negated gradient field does not have any local minima. This closed-loop approach guarantees the convergence to a desired destination, as well as collision avoidance.

In [4], the navigation function framework is extended to multi-agent system. In [5], a centralized navigation function control strategy is proposed to steer a group of mobile agents to achieve a particular formation while avoiding collisions between the agents, and obstacles in the environment. A limited sensing region of agents is assumed in [5] for obstacle avoidance assuming that the agents are connected. The problem of connectivity maintenance is addressed using a centralized navigation function in [6] and [7] by modeling the sensing region of the base station as a workspace and assuming that agents are always connected to the base station as long as they stay within the workspace. In [8], a decentralized navigation function is developed to achieve obstacle avoidance assuming that the agents are connected. A decentralized navigation function is also used in [9] for motion control of agents with global knowledge of the position of other agents. For agents with limited sensing capabilities, a decentralized navigation function approach is presented in [10] with the assumption that total number of agents in the system is apriori known. In [11], a navigation function based path planning algorithm is developed for multiple UAVs with finite sensing capabilities in a combat area.

A review of literature indicates that most formation control efforts simply assume that agents within the network are able to communicate, such as [5] and [8]. The assumption of a connected graph is restrictive in the case of a mobile network, where communication between a pair of agents depends (in part) on the distance between agents. In practical applications, each agent has limited communication and sensing capability to determine relative position and velocity information that would be required by the agent's control system. Thus, connectivity maintenance is a key requirement for formation control.

For multi-agent systems (especially for a large number of agents), a centralized control approach has a higher cost to implement than a decentralized approach because of the increased computation load and the decreased robustness [12]. However, since the motion by any agent in the network may partition the underlying network graph (i.e., connectivity is a global graph property [13]), maintaining connectivity of a formation is challenging for a decentralized control scheme that only relies on locally available information. In a related work [14], a potential field is designed for a group of mobile agents to form a desired configuration while maintain network connectivity. However, the mission maybe fail because of the existence of local minima. The contribution of this paper is the development of a decentralized control scheme for each agent that ensures network connectivity and achieves a desired formation using only local information. Specifically, by using a navigation function formulation, the developed decentralized

[1]This research is supported by National Science Foundation grant number CNS-0626863.

[2]Z. Kan, A. P. Dani and W. E. Dixon are with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611-6250, USA. W. E. Dixon is also with the Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611-6250, USA. Email: {kanzhen0322, ashwin31, wdixon}@ufl.edu

[3]J.M. Shea is with the Department of Electrial and Computer Engineering, University of Florida, Gainesville, FL 32611-6250, USA. Email: jshea@ece.ufl.edu

controller enables an initially connected multi-agent system to achieve a desired formation, as well as avoid collision with other agents and obstacles in the environment while ensuring the underlying network graph does not partition. Preliminary simulation results illustrate the performance of the developed controller.

II. Problem Formulation

Consider a network composed of N agents in the workspace $\mathcal{F}$, where agent i moves according to the following kinematics:

$$\dot{q}_i = u_i,\ i = 1, \cdots, N \tag{1}$$

where $q_i \in \mathbb{R}^2$ denotes the position of agent i in a two dimensional (2D) plane, and $u_i \in \mathbb{R}^2$ denotes the velocity of agent i (i.e., the control input). The workspace $\mathcal{F}$ is assumed to be circular and bounded with radius R, and $\partial\mathcal{F}$ denotes the boundary of $\mathcal{F}$. Each agent in $\mathcal{F}$ is represented by a point-mass with a limited communication and sensing capability encoded by a disk area. Two moving agents can communicate with each other if they stay within a disk area with radius R_c and obstacles can be sensed whenever they enter the sensing area. For simplicity and without loss of generality, the following development is based on the assumption that the sensing area is the same as the communication area, both with radius R_c. Further, it is assumed that all the agents have equal actuation capabilities. A set of fixed points, $p_1, \cdots, p_M$, are defined to represent M stationary obstacles in the workspace $\mathcal{F}$, and the index set of obstacles is denoted as $\mathcal{M} = \{1, \cdots, M\}$.

The interaction of the multi-agent system is modeled as a undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with the set of nodes $\mathcal{V} = \{1, \cdots, N\}$ and the set of edges $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} | \, d_{ij} \leq R_c\}$, where node i and node j are located at a position q_i and q_j, and $d_{ij} \in \mathbb{R}^+$ is the distance between them defined as $d_{ij} = \|q_i - q_j\|$. In graph $\mathcal{G}$, each node i denotes an agent i and each edge (i, j) denotes a communication link between agent i and j when they stay within each other's communication area. It is assumed that each agent has real time knowledge of its own position. The set of neighbors of node i (i.e., all the agents within the sensing zone of agent i) is given by $\mathcal{N}_i = \{j,\ j \neq i | \, j \in \mathcal{V},\ (i, j) \in \mathcal{E}\}$. One objective for the multi-agent system in this work is to converge to a desired configuration, which is determined by a formation matrix $C \in \mathbb{R}^{2N \times N}$, where each parameter $c_{ij} \in \mathbb{R}^2$ represents the desired relative position and orientation of node i with an adjacent node $j \in \mathcal{N}_i^f$, where $\mathcal{N}_i^f \subset \mathcal{N}_i$ denotes a neighborhood that includes only nodes in the desired configuration. The neighborhood $\mathcal{N}_i$ is a time varying set since nodes may enter or leave the communication region of node i at any time instant, while $\mathcal{N}_i^f$ is a static set which is specified by the desired configuration. The desired position of node i, denoted by q_{di}, is defined as

$$q_{di} = \left\{ q_i | \, \|q_i - q_j - c_{ij}\|^2 = 0,\ j \in \mathcal{N}_i^f \right\}. \tag{2}$$

An edge (i, j) is only established between nodes i and j if $j \in \mathcal{N}_i^f$.

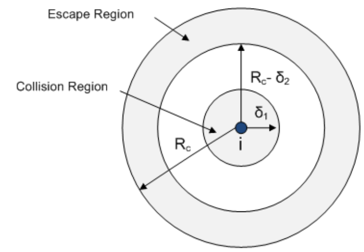

Fig. 1. Collision and escape regions for node i.

A *collision region*[1] defined for each agent i is a small disk around the agent i with radius $\delta_1 < R_c$, such that any other agent $j \in \mathcal{N}_i$, or obstacle p_k, $k \in \mathcal{M}$, inside this region is considered as a potential collision with agent i. To ensure connectivity, an *escape region* for each agent i is defined as the outer ring of the communication area with radius r, $R_c - \delta_2 < r < R_c$, where $\delta_2 \in \mathbb{R}$ is a predetermined buffer distance. Edges formed with any node $j \in \mathcal{N}_i^f$ in the escape region are in danger of breaking. The collision and escape regions for node i are shown in Fig. 1.

The goal in this paper is to develop a decentralized controller u_i that will ensure network connectivity and enable the system to stabilize in a desired configuration from almost all initial conditions, except for a set of measure zero points (i.e., the saddle points), while avoiding collisions among themselves, as well as obstacles. To achieve this goal, the subsequent development is based on the following assumptions.

Assumption 1: The initial graph $\mathcal{G}$ is valid in the sense that the initial network is connected with the desired edge neighborhood (i.e., any node $j \in \mathcal{N}_i^f$ is connected to node i initially)[2] and those initial positions do not coincide with some unstable equilibria (i.e., saddle points).

Assumption 2: The desired formation matrix C is specified initially and is achievable. This assumption implies that the desired configuration will not lead to a collision or the desired configuration will not lead to a partitioned graph, (i.e., $\delta_1 < \|c_{ij}\| < R_c - \delta_2$).

III. Control Design

Consider a decentralized navigation function candidate $\varphi_i : \mathcal{F} \to [0, 1]$ for each node i as

$$\varphi_i = \frac{\gamma_i}{(\gamma_i^\alpha + \beta_i)^{1/\alpha}}, \tag{3}$$

where $\alpha \in \mathbb{R}^+$ is a tuning parameter, $\gamma_i : \mathbb{R}^2 \to \mathbb{R}^+$ is the goal function, and $\beta_i : \mathbb{R}^2 \to [0, 1]$ is a constraint function for node i. The goal function γ_i in (3) encodes the control

[1] The potential collision for node i in this work not only refers to the fixed obstacles, but also other moving nodes or the workspace boundary, which are currently located in its collision region.

[2] If the network is initially connected but not in the desired edge neighborhood, then recent methods such as in [15] can be used to reorganize the network to the desired edge neighborhood while maintaining network connectivity.

objective of node i and is designed as

$$\gamma_i(q_i) = \sum_{j\in\mathcal{N}_i^f} \|q_i - q_j - c_{ij}\|^2. \tag{4}$$

The constraint function β_i in (3) is designed as

$$\beta_i = B_{i0} \prod_{j\in\mathcal{N}_i^f} b_{ij} \prod_{k\in\mathcal{N}_i\cup\mathcal{M}_i} B_{ik}, \tag{5}$$

to ensure collision avoidance and network connectivity by only accounting for nodes and obstacles located within its sensing area each time instant. In (5), $b_{ij} \triangleq b(q_i, q_j) : \mathbb{R}^2 \to [0, 1]$ ensures connectivity of the network graph (i.e., guarantees that node $j \in \mathcal{N}_i^f$ will never leave the communication zone of node i if node j is initially connected to node i) and is designed as

$$b_{ij} = \begin{cases} 1 & d_{ij} \leq R_c - \delta_2 \\ -\frac{1}{\delta_2^2}(d_{ij} + 2\delta_2 - R_c)^2 \\ +\frac{2}{\delta_2}(d_{ij} + 2\delta_2 - R_c) & R_c - \delta_2 < d_{ij} < R_c \\ 0 & d_{ij} \geq R_c. \end{cases} \tag{6}$$

Also in (5), $B_{ik} \triangleq B(q_i, q_k) : \mathbb{R}^2 \to [0, 1]$, for point $k \in \mathcal{N}_i \cup \mathcal{M}_i$, where $\mathcal{M}_i$ indicates the set of obstacles within the sensing area of node i at each time instant, ensures that node i is repulsed from other nodes or obstacles to prevent a collision, and is designed as

$$B_{ik} = \begin{cases} -\frac{1}{\delta_1^2}d_{ik}^2 + \frac{2}{\delta_1}d_{ik} & d_{ik} < \delta_1 \\ 1 & d_{ik} \geq \delta_1. \end{cases} \tag{7}$$

Similarly, the function B_{i0} in (5) is used to model the potential collision of node i with the workspace boundary, where the positive scalar $B_{i0} \in \mathbb{R}$ is designed as

$$B_{i0} = \begin{cases} -\frac{1}{\delta_1^2}d_{i0}^2 + \frac{2}{\delta_1}d_{i0} & d_{i0} < \delta_1 \\ 1 & d_{i0} \geq \delta_1, \end{cases} \tag{8}$$

where $d_{i0} \in \mathbb{R}^+$ is the relative distance of the node i to the workspace boundary defined as $d_{i0} = R - \|q_i\|$.

Assumption 2 guarantees that γ_i and β_i will not be zero simultaneously. The navigation function candidate achieves its minimum of 0 when $\gamma_i = 0$ and achieves its maximum of 1 when $\beta_i = 0$. For φ_i to be a navigation function, it has to satisfy the following conditions [2]: 1) smooth on $\mathcal{F}$ (at least a $\mathcal{C}^2$ function); 2) admissible on $\mathcal{F}$, (uniformly maximal on $\partial\mathcal{F}$ and constraint boundary); 3) polar on $\mathcal{F}$, (q_{di} is a unique minimum); 4) a Morse function, (critical points[3] of the navigation function are non-degenerate).

If φ_i is a Morse function and q_{di} is a unique minimum of φ_i (i.e., q_{di} is polar on $\mathcal{F}$), then almost all initial positions (except for a set of measure zero points) asymptotically approach the desired position q_{di} [2]. In addition, the negative gradient of the navigation function is bounded if it is an admissible Morse function with a single minimum at the desired destination [2].

Based on the definition of the navigation function candidate, the decentralized controller for each node is designed as

$$u_i = -K_i \nabla_{q_i} \varphi_i, \tag{9}$$

[3]A point p in the workspace $\mathcal{F}$ is a critical point if $\nabla_{q_i}\varphi_i|_p = 0$.

where K_i is a positive gain, and $\nabla_{q_i}\varphi_i$ is the gradient of the φ_i with respect to q_i. Hence, the controller in (9) is bounded and yields the desired performance by steering node i along the direction of the negative gradient of φ_i if (3) is a navigation function.

IV. Connectivity and Convergence Analysis

The free configuration workspace $\mathcal{F}_i \subset \mathcal{F}$ is a compact connected analytic manifold for node i, $\mathcal{F}_i \triangleq \{\mathbf{q} | \beta_i(\mathbf{q}) > 0\}$, and $\mathbf{q}$ denotes the stacked position vector of node i. The boundary of $\mathcal{F}_i$ is defined as $\partial\mathcal{F}_i \triangleq \beta_i^{-1}(0)$. The narrow set around a potential collision for node i is defined as $\mathcal{B}_{i,k}^B(\varepsilon) \triangleq \{\mathbf{q} | 0 < B_{ik} < \varepsilon, \varepsilon > 0, k \in \mathcal{N}_i \cup \mathcal{M}_i\}$ and a narrow set around a potential connectivity constraint is defined as $\mathcal{B}_{i,j}^b(\varepsilon) \triangleq \{\mathbf{q} | 0 < b_{ij} < \varepsilon, \varepsilon > 0, j \in \mathcal{N}_i^f\}$. The set $\mathcal{B}_0(\varepsilon) = \{\mathbf{q} | 0 < B_{i0} < \varepsilon, \varepsilon > 0\}$ is used to denote a narrow set around a potential collision of node i with workspace boundary. Inspired by the seminal work [2], $\mathcal{F}_i$ is partitioned into five subsets $\mathcal{F}_0(\varepsilon)$, $\mathcal{F}_1(\varepsilon)$, $\mathcal{F}_2(\varepsilon)$, $\mathcal{F}_3(\varepsilon)$, and $\mathcal{F}_{di}(\varepsilon)$ as

$$\mathcal{F}_i = \mathcal{F}_{di} \cup \mathcal{F}_0(\varepsilon) \cup \mathcal{F}_1(\varepsilon) \cup \mathcal{F}_2(\varepsilon) \cup \mathcal{F}_3(\varepsilon). \tag{10}$$

In (10), the set of desired configurations for node i is defined as $\mathcal{F}_{di} \triangleq \{\mathbf{q} | \gamma_i(\mathbf{q}) = 0\}$. Similar to [2], the sets $\mathcal{F}_0(\varepsilon)$, $\mathcal{F}_1(\varepsilon)$, $\mathcal{F}_2(\varepsilon)$ and $\mathcal{F}_3(\varepsilon)$ describe the regions near the workspace boundary, near the potential collision constraint, near the connectivity constraint and away from all constraint for node i, respectively. Based on the partition of $\mathcal{F}_i$ in (10), *Proposition 1-7* are introduced to ensure that the designed function in (3) is a navigation function. The following Assumptions are used to prove the *Propositions*.

Assumption 3: There are no obstacles or other agents that stay within the collision region of node i, when node i is very close to breaking the communication link with a node $j \in \mathcal{N}_i^f$ (i.e., node i and node j belong to the region $\mathcal{B}_{i,j}^b(\varepsilon)$).

Assumption 4: The region $\mathcal{B}_{i,k}^B(\varepsilon)$ for $k \in \mathcal{N}_i \cup \mathcal{M}_i$ is disjoint. This assumption implies that the probability of more than one collision with node i simultaneously is negligible.

A. Connectivity Analysis

Proposition 1: If the graph $\mathcal{G}$ is connected initially and $j \in \mathcal{N}_i^f$, then nodes i and j will remain connected for all the future time under the control law (9).

Proof: Consider node i located at a point $q_0 \in \mathcal{F}$ that causes $\prod_{j\in\mathcal{N}_i^f} b_{ij} = 0$, which will be true when either only one node j is about to disconnect from node i or when more than one node is about to disconnect with node i simultaneously. These two possibilities are considered in following two cases.

Case 1: There is only one node $j \in \mathcal{N}_i^f$ for which $b_{ij}(q_0, q_j) = 0$ and $b_{il}(q_0, q_l) \neq 0\ \forall l \in \mathcal{N}_i^f$, $l \neq j$. The gradient of φ_i with respect to q_i is

$$\nabla_{q_i}\varphi_i = \frac{\alpha\beta_i\nabla_{q_i}\gamma_i - \gamma_i\nabla_{q_i}\beta_i}{\alpha(\gamma_i^\alpha + \beta_i)^{\frac{1}{\alpha}+1}}. \tag{11}$$

Since $b_{ij} = 0$, the constraint function $\beta_i = 0$ from (5). Thus, the gradient $\nabla_{q_i}\varphi_i$ evaluated at q_0 can be expressed as

$$\nabla_{q_i}\varphi_i|_{q_0} = -\left.\frac{\gamma_i \nabla_{q_i}\beta_i}{\alpha\gamma_i^{\alpha+1}}\right|_{q_0}. \quad (12)$$

Based on the fact that β_i can be expressed as the product $\beta_i = b_{ij}\bar{b}_{ij}$, where

$$\bar{b}_{ij}(q_0, q_j) = B_{i0} \prod\nolimits_{l\in\mathcal{N}_i^f, l\neq j} b_{il} \prod\nolimits_{k\in\mathcal{N}_i\cup\mathcal{M}_i} B_{ik}, \quad (13)$$

and $\nabla_{q_i} b_{ij}$ is computed as

$$\nabla_{q_i} b_{ij} = \begin{cases} 0 & d_{ij} < R_c - \delta_2 \\ & \text{or } d_{ij} > R_c \\ -\frac{2(d_{ij}+\delta_2-R_c)(q_i-q_j)}{\delta_2^2 d_{ij}} & R_c - \delta_2 \leq d_{ij} \leq R_c, \end{cases} \quad (14)$$

the gradient of β_i evaluated at q_0 can be obtained as

$$\nabla_{q_i}\beta_i|_{q_0} = -\frac{2\bar{b}_{ij}}{\delta R_c}(q_i - q_j). \quad (15)$$

Since K_i, γ_i, α, $\bar{b}_{ij}$ and δ are all positive terms, (9), (12), and (15) can be used to determine that the controller (i.e., the negative gradient of $\nabla_{q_i}\varphi_i$) is along the direction of $q_j - q_i$, which implies node i is forced to move toward node j to ensure connectivity. That is, based on the design of b_{ij} in (6) and its gradient in (14), whenever a node enters the escape region of node i, an attractive force is imposed on node i to ensure connectivity.

Case 2[4]: Consider two nodes $j, l \in \mathcal{N}_i^f$, where $b_{ij} = 0$ and $b_{il} = 0$ (i.e., $\|q_i - q_j\| = R_c$ and $\|q_i - q_l\| = R_c$) simultaneously. In this case, $\beta_i = 0$ and $\nabla_{q_i}\beta_i$ is a zero vector, (11) can be used to determine that q_0 is a critical point (i.e., $\nabla_{q_i}\varphi_i|_{q_0} = 0$), and the navigation function achieves its maximum value at the critical point (i.e., $\varphi_i|_{q_0} = 1$). Since φ_i is maximized at q_0 no open set of initial conditions can be attracted to q_0 under the control law designed in (9).

From the development in Case 1 and Case 2, the control law in (9) ensures that all nodes $j \in \mathcal{N}_i^f$ remain connected with node i for all time. ∎

B. Convergence Analysis

Proposition 2: The navigation function is minimized at the desired point q_{di}.

Proof: The navigation function φ_i is minimized at a critical point if the Hessian of φ_i evaluated at that point is positive definite. From (2) and (4), the goal function evaluated at the desired point is $\gamma_i|_{q_{di}} = 0$. Also, the gradient of the goal function evaluated at the desired point q_{di} is $\nabla_{q_i}\gamma_i|_{q_{di}} = \sum_{j\in\mathcal{N}_i^f} 2(q_{di} - q_j - c_{ij}) = 0$. Since $\gamma_i|_{q_{di}} = 0$ and $\nabla_{q_i}\gamma_i|_{q_{di}} = 0$, (11) can be used to conclude that $\nabla_{q_i}\varphi_i|_{q_{di}} = 0$. Thus, the desired point q_{di} in the workspace $\mathcal{F}$ is a critical point of φ_i. The Hessian of φ_i is

$$\begin{aligned}\nabla_{q_i}^2\varphi_i = & \frac{1}{\alpha(\gamma_i^\alpha + \beta_i)^{\frac{1}{\alpha}+2}} \Big\{ (\gamma_i^\alpha + \beta_i) \left[\alpha\nabla_{q_i}\beta_i (\nabla_{q_i}\gamma_i)^T \right. \\ & \left. -\nabla_{q_i}\gamma_i(\nabla_{q_i}\beta_i)^T + \alpha\beta_i\nabla_{q_i}^2\gamma_i - \gamma_i\nabla_{q_i}^2\beta_i\right] \\ & -\frac{\alpha+1}{\alpha}\left[\alpha\beta_i\nabla_{q_i}\gamma_i - \gamma_i\nabla_{q_i}\beta_i\right] \\ & \cdot\left[\alpha\gamma_i^{\alpha-1}\nabla_{q_i}\gamma_i + \nabla_{q_i}\beta_i\right]^T\Big\}. \end{aligned} \quad (16)$$

Using the facts that $\gamma_i|_{q_{di}} = 0$ and $\nabla_{q_i}\gamma_i|_{q_{di}} = 0$ and the Hessian of γ_i is

$$\nabla_{q_i}^2\gamma_i = 2\zeta_i I_2, \quad (17)$$

where I_2 is the identity matrix in $\mathbb{R}^{2\times2}$, the Hessian of φ_i evaluated at q_{di} is given by $\nabla_{q_i}^2\varphi_i|_{q_{di}} = 2\beta_i^{-\frac{1}{\alpha}} I_2\zeta_i$. The constraint function $\beta_i > 0$ at the desired configuration by Assumption 2, and ζ_i is a positive number. Hence, the Hessian of φ_i evaluated at that point is positive definite. ∎

Proposition 3: No minimum of φ_i are on the boundary of the free workspace $\mathcal{F}_i$.

Proof: Consider a point $q_0 \in \partial\mathcal{F}_i$. From the definition of $\partial\mathcal{F}_i$ the constraint function $\beta_i(q_0) = 0$. The goal function γ_i is zero only at the desired configuration point, and from Assumption 2, the desired configuration cannot be on the boundary of $\mathcal{F}_i$. Thus, the goal function γ_i evaluated at q_0 is not zero. Using (3) and the facts that $\beta_i|_{q_0} = 0$ and $\gamma_i|_{q_0} \neq 0$, $\varphi_i|_{q_0}$ is maximized at any arbitrarily chosen point q_0 on the boundary of $\mathcal{F}_i$. ∎

Proposition 4: For every $\varepsilon > 0$, there exists a number $\Gamma(\varepsilon)$ such that if $\alpha > \Gamma(\varepsilon)$ no critical points of φ_i are in $\mathcal{F}_3(\varepsilon)$.

Proof: From (11), any critical point must satisfy

$$\alpha\beta_i\nabla_{q_i}\gamma_i = \gamma_i\nabla_{q_i}\beta_i. \quad (18)$$

If α is chosen as

$$\alpha > \sup \frac{\gamma_i \|\nabla_{q_i}\beta_i\|}{\beta_i \|\nabla_{q_i}\gamma_i\|}, \quad (19)$$

where sup is taken over $\mathcal{F}_3(\varepsilon)$, then from (11) and (18), φ_i will have no critical points in $\mathcal{F}_3(\varepsilon)$. Since $\varepsilon = \inf b_{ij} = \inf B_{ik}$ in $\mathcal{F}_3(\varepsilon)$, an upper bound for the right hand side of (19) is given as $\Gamma(\varepsilon)$, where

$$\begin{aligned}\Gamma(\varepsilon) \triangleq & \sup\frac{\gamma_i}{\|\nabla_{q_i}\gamma_i\|}\left(\sum\nolimits_{j=1,j\neq i}^{\zeta_i} \frac{\sup\|\nabla_{q_i}b_{ij}\|}{\varepsilon}\right. \\ & \left. + \sum\nolimits_{k=0,k\neq i}^{\xi_i+\vartheta_i}\frac{\sup\|\nabla_{q_i}B_{ik}\|}{\varepsilon}\right). \end{aligned} \quad (20)$$

In (20), $\|\nabla_{q_i}b_{ij}\|$, $\|\nabla_{q_i}B_{ik}\|$ and $\frac{\gamma_i}{\|\nabla_{q_i}\gamma_i\|}$ are bounded terms in $\mathcal{F}_3(\varepsilon)$ from (4), (14) and the fact that

$$\nabla_{q_i}B_{ik} = \begin{cases} (-\frac{2}{\delta_1^2}d_{ik} + \frac{2}{\delta_1})\frac{q_i - q_k}{d_{ik}} & d_{ik} < \delta_1 \\ 0 & d_{ik} \geq \delta_1. \end{cases} \quad (21)$$

∎

Proposition 5: There exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then φ_i is a Morse function.

Proof: The development in [3] and [5] proves that for φ_i to be a Morse function, it is sufficient to show that

[4]Case 2 can be extended to more than two nodes without loss of generality.

$\hat{u}^T(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}})\hat{u}$ is positive for some particular vector $\hat{u}$ by choosing a small ε, where q_{ci} is a critical point. To show that $\hat{u}^T(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}})\hat{u}$ is positive for the unit vector $\hat{u} \triangleq \frac{q_i - q_j}{\|q_i - q_j\|}$, (16) is used and the Hessian $\nabla^2_{q_i}\varphi_i$ evaluated at q_{ci} is

$$\frac{\alpha\hat{u}^T\left(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}\right)\hat{u}}{(\gamma_i^\alpha+\beta_i)^{-\frac{1}{\alpha}-1}} = \hat{u}^T\left(\alpha\beta_i\nabla^2_{q_i}\gamma_i + \frac{(1-\frac{1}{\alpha})\gamma_i}{\beta_i} \cdot\nabla_{q_i}\beta_i\left(\nabla_{q_i}\beta_i\right)^T - \gamma_i\nabla^2_{q_i}\beta_i\right)\hat{u}. \tag{22}$$

To facilitate the subsequent analysis, the set of critical points in $\mathcal{F}_i$ is divided into sets of critical points in regions $\mathcal{F}_0(\varepsilon)$, $\mathcal{F}_1(\varepsilon)$, and $\mathcal{F}_2(\varepsilon)$. For a case where a critical point $q_{ci} \in \mathcal{F}_2(\varepsilon)$, using the fact that the first term on the right hand side of (22) is always positive from (17), the subsequent expression can be obtained as

$$\alpha(\gamma_i^\alpha+\beta_i)^{\frac{1}{\alpha}+1}\hat{u}^T\left(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}\right)\hat{u} > \gamma_i\Omega, \tag{23}$$

where

$$\Omega = \frac{1}{b_{ij}}\left(a_1 b_{ij}^2 + a_2 b_{ij} + a_3\right), \tag{24}$$

where

$$\begin{aligned} a_1 &= \frac{(\alpha-1)\left\|\nabla_{q_i}\bar{b}_{ij}\right\|^2}{\alpha\bar{b}_{ij}} - \hat{u}^T(\nabla^2_{q_i}\bar{b}_{ij})\hat{u}, \qquad (25)\\ a_2 &= \frac{2(\alpha-1)(\nabla_{q_i}\bar{b}_{ij})^T(\nabla_{q_i}b_{ij})}{\alpha} - \bar{b}_{ij}\hat{u}^T(\nabla^2_{q_i}b_{ij})\hat{u} \\ &\quad -\hat{u}^T\left(\nabla_{q_i}\bar{b}_{ij}\nabla^T_{q_i}b_{ij} + \nabla_{q_i}b_{ij}\nabla^T_{q_i}\bar{b}_{ij}\right)\hat{u},\\ a_3 &= \frac{(\alpha-1)\bar{b}_{ij}}{\alpha}\left\|\nabla_{q_i}b_{ij}\right\|^2. \end{aligned}$$

Since $b_{ij} > 0$, a necessary condition to show that $\Omega > 0$ is to prove that

$$a_1 b_{ij}^2 + a_2 b_{ij} + a_3 > 0, \tag{26}$$

where $a_3 > 0$ if $\alpha > 1$. To prove the inequality in (26), the following two cases are analyzed.

Case 1: For $a_1 < 0$, the inequality in (26) is valid if

$$b_{ij} < \frac{-a_2 - \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}. \tag{27}$$

Case 2: For $a_1 \geq 0$, Ω can be rewritten as

$$\Omega \geq a_2 + \frac{a_3}{b_{ij}}, \tag{28}$$

which is positive if $b_{ij} < \frac{a_3}{|a_2|}$.

Therefore, $\Omega > 0$, and from (23), $\hat{u}^T(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}})\hat{u} > 0$ for all cases if b_{ij} is chosen as

$$b_{ij} < \varepsilon_0^{'} \triangleq \min\left\{\frac{-a_2 - \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}, \frac{a_3}{|a_2|}\right\}. \tag{29}$$

By using the same process of evaluating the Hessian $\nabla^2_{q_i}\varphi_i$ at critical points belonging to $\mathcal{F}_0(\varepsilon)$ and $\mathcal{F}_1(\varepsilon)$, upper bounds $\varepsilon_0^{''}$ and $\varepsilon_0^{'''}$ for ε can be obtained for $q_{ci} \in \mathcal{F}_1(\varepsilon)$ and $q_{ci} \in \mathcal{F}_0(\varepsilon)$ respectively. By choosing $\varepsilon < \varepsilon_0 = \min\left\{\varepsilon_0^{'}, \varepsilon_0^{''}, \varepsilon_0^{'''}\right\}$, the function Ω in (24) is guaranteed to be positive which implies all the critical points are non-degenerate critical points of φ_i. ■

Proposition 6: There exists $\varepsilon_1 > 0$, such that φ_i has no local minimum in $\mathcal{F}_2(\varepsilon)$, as long as $\varepsilon < \varepsilon_1$.

Proof: Consider a critical point $q_{ci} \in \mathcal{F}_2(\varepsilon)$. Since φ_i is a Morse function, then if $\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}$ has at least one negative eigenvalue, φ_i will have no minimum in $\mathcal{F}_2(\varepsilon)$. To show $\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}$ has at least one negative eigenvalue, a unit vector $\hat{v} \triangleq \left(\frac{\nabla_{q_i}\beta_i}{\|\nabla_{q_i}\beta_i\|}\right)^\perp$ is defined as a test direction to demonstrate that $\hat{v}^T\left(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}\right)\hat{v} < 0$, where $(\chi)^\perp$ denotes a vector that is perpendicular to some vector χ. From *Proposition 5,*

$$\alpha(\gamma_i^\alpha+\beta_i)^{\frac{1}{\alpha}+1}\Big|_{q_{ci}}\hat{v}^T\left(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}\right)\hat{v} = -\gamma_i\Phi + b_{ij}\Psi, \tag{30}$$

where

$$\Phi = \hat{v}^T\left(\nabla_{q_i}\bar{b}_{ij}\nabla^T_{q_i}b_{ij} + \nabla_{q_i}b_{ij}\nabla^T_{q_i}\bar{b}_{ij} - \bar{b}_{ij}\nabla^2_{q_i}b_{ij}\right)\hat{v}, \tag{31}$$

$$\Psi = \hat{v}^T\left(\alpha\bar{b}_{ij}\nabla^2_{q_i}\gamma_i - \gamma_i\nabla^2_{q_i}\bar{b}_{ij}\right)\hat{v}, \tag{32}$$

Based on Assumption 3 and (6), (7), (14),

$$\nabla_{q_i}\bar{b}_{ij} = 0 \text{ and } \nabla^2_{q_i}b_{ij} < 0. \tag{33}$$

Since the goal function γ_i and $\bar{b}_{ij}$ are positive, $\Phi > 0$. To ensure $\hat{v}^T\left(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}\right)\hat{v} < 0$, ε must be selected as $\varepsilon < \varepsilon_1$ where $\varepsilon_1 = \inf_{\mathcal{F}_2(\varepsilon)}\frac{|\gamma_i\Phi|}{|\Psi|}$. ■

Proposition 7: There exists $\varepsilon_2 > 0$, such that φ_i has no local minimum in $\mathcal{F}_1(\varepsilon)$ and $\mathcal{F}_0(\varepsilon)$, as long as $\varepsilon < \varepsilon_2$.

Proof: Consider a critical point $q_{ci} \in \mathcal{F}_1(\varepsilon)$. Similar to the proof for *Proposition 6*, the current proof is based on the fact that if $\hat{w}^T\left(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}\right)\hat{w} < 0$ for some particular vector $\hat{w} \triangleq \left(\frac{q_i - q_k}{\|q_i - q_k\|}\right)^\perp$, then φ_i will have no minimum in $\mathcal{F}_1(\varepsilon)$. To facilitate the subsequent analysis, similar to the definition of $\bar{b}_{ij}$ in (13), β_i can be expressed as the product $\beta_i = B_{ik}\bar{B}_{ik}$ and $\bar{B}_{ik}$ is defined as

$$\bar{B}_{ik}(q_i, q_k) = B_{i0}\prod_{j\in\mathcal{N}_i^f} b_{ij}\prod_{l\in\mathcal{N}_i\cup\mathcal{M}_i, l\neq k} B_{il}. \tag{34}$$

Using (17), (21) and (34),

$$\alpha(\gamma_i^\alpha+\beta_i)^{\frac{1}{\alpha}+1}\Big|_{q_{ci}}\hat{w}^T\left(\nabla^2_{q_i}\varphi_i\big|_{q_{ci}}\right)\hat{w} = \gamma_i\bar{B}_{ik}\Lambda + \gamma_i B_{ik}\Xi, \tag{35}$$

where

$$\Lambda = \nabla^T_{q_i}B_{ik}\frac{\nabla_{q_i}\gamma_i}{\|\nabla_{q_i}\gamma_i\|}2\zeta_i - \frac{2(\delta_1 - d_{ik})}{d_{ik}\delta_1^2} \tag{36}$$

$$\Xi = \hat{w}^T\left(\frac{\nabla^T_{q_i}\bar{B}_{ik}\nabla_{q_i}\gamma_i}{\|\nabla_{q_i}\gamma_i\|}\nabla^2_{q_i}\gamma_i + \frac{(1-\frac{1}{\alpha})}{\bar{\beta}_{ij}}\nabla_{q_i}\bar{B}_{ik}\nabla^T_{q_i}\bar{B}_{ik} - \nabla^2_{q_i}\bar{B}_{ik}\right)\hat{w}, \tag{37}$$

Since $d_{ik} < \delta_1$, and $\nabla^T_{q_i}B_{ik}\frac{\nabla_{q_i}\gamma_i}{\|\nabla_{q_i}\gamma_i\|}$ can be upper bounded by a positive constant in $\mathcal{F}_1(\varepsilon)$, then if d_{ik} is small enough,

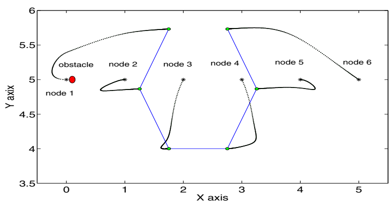

Fig. 2. Trajectory evolution for each node.

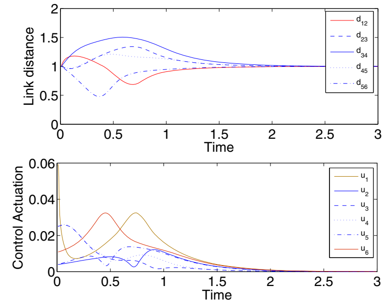

Fig. 3. Distance between connected nodes and control actuation for each node.

Λ is guaranteed to be negative. Hence, there exist a positive scalar $\varepsilon_{21} = B_{ik}(d_{ik})$, which is small enough to ensure $\Lambda < 0$. To ensure $\hat{w}^T \left(\nabla^2_{q_i} \varphi_i \big|_{q_{ci}} \right) \hat{w} < 0$, ε must be selected as $\varepsilon < \min\{\varepsilon_{21}, \inf_{\mathcal{F}_1(\varepsilon)} \frac{|\Lambda \bar{B}_{ik}|}{|\Xi|}\}$.

Let $\hat{x}$ be an unit vector defined as $\hat{x} \triangleq \left(\frac{q_i - q_0}{\|q_i - q_0\|} \right)^{\perp}$. The same procedure that was used to show $\hat{w}^T \left(\nabla^2_{q_i} \varphi_i \big|_{q_{ci}} \right) \hat{w} < 0$ in $\mathcal{F}_1(\varepsilon)$ can be followed to obtain another upper bound for ε, which ensures $\hat{x}^T \left(\nabla^2_{q_i} \varphi_i \big|_{q_{ci}} \right) \hat{x} < 0$ in $\mathcal{F}_0(\varepsilon)$. By choosing ε_2 as the minimum of the upper bound for ε developed for $\mathcal{F}_1(\varepsilon)$ and $\mathcal{F}_0(\varepsilon)$, φ_i is ensured to have no minimum in $\mathcal{F}_1(\varepsilon)$ and $\mathcal{F}_0(\varepsilon)$ as long as $\varepsilon < \varepsilon_2$. ■

Based on *Propositions 4-7*, if ε is chosen such that $\varepsilon \leq \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ then the minima (a critical point) is not in $\mathcal{F}_0(\varepsilon)$, $\mathcal{F}_1(\varepsilon)$, $\mathcal{F}_2(\varepsilon)$, $\mathcal{F}_3(\varepsilon)$ or the boundary of $\mathcal{F}_i$. Thus, from (10) the minima has to be in $\mathcal{F}_{di}(\varepsilon)$ if $\alpha > \max\{1, \Gamma(\varepsilon)\}$. Hence, nodes starting from any initial positions (except for the unstable equilibria) will converge to the desired formation specified by the formation matrix C.

V. Simulation

A group of 6 nodes with kinematics given in (1) are distributed in a workspace of $R = 100\ m$. Each node is assumed to have a limited communication and sensing zone of $R_c = 2\ m$ and $\delta_1 = \delta_2 = 0.4\ m$. The tuning parameter α in (3) is set as $\alpha = 1.5$. The system is simulated for $3s$ with the step size of 0.01. The simulation results are shown in Figs. 2 and 3. In Fig. 2, the '*' represents the initial position of each node, and the circle denotes its final position. The initial graph is connected and the desired configuration is characterized as a regular hexagon with edge of $1\ m$. A stationary obstacle is represented by a red circle. The trajectory of each node is represented by a dotted curve connecting its initial and final position while the solid lines indicate the communication link between connected nodes. As shown in Fig. 2, the system finally converges to the desired configuration. The first subplot in Fig. 3 shows that the control actuation for each node is bounded while the second subplot in Fig. 3 indicates that the communication link is maintained during the evolution (i.e., the distance between connected nodes is less than R_c).

VI. Conclusion

A decentralized navigation function is developed to ensure that a network that is initially connected with the desired edge neighborhood will asymptotically converge to the desired configuration while maintain network connectivity from almost all initial positions, as well as collision avoidance.

References

[1] D. Dimarogonas and K. Johansson, "Analysis of robot navigation schemes using Rantzer dual Lyapunov theorem," in *Proc. IEEE Am. Control Conf.*, june 2008, pp. 201 –206.

[2] E. Rimon and D. Koditschek, "Exact robot navigation using artificial potential functions," *IEEE Trans. Robot. Autom.*, vol. 8, no. 5, pp. 501–518, Oct 1992.

[3] D. E. Koditschek and E. Rimon, "Robot navigation functions on manifolds with boundary," *Adv. Appl. Math.*, vol. 11, pp. 412–442, Dec 1990.

[4] S. Loizou and K. Kyriakopoulos, "Closed loop navigation for multiple holonomic vehicles," in *Proc. IEEE/RSJ Int. Conf. Intell. Robot. Syst.*, vol. 3, 2002, pp. 2861 – 2866.

[5] H. Tanner and A. Kumar, "Towards decentralization of multi-robot navigation functions," in *Proc. IEEE Int. Conf. Robot. Autom.*, April 2005, pp. 4132 – 4137.

[6] A. Ghaffarkhah and Y. Mostofi, "Communication-aware target tracking using navigation functions - centralized case," in *Int. Conf. Robot Commun. Co-ord.*, March 31 - April 2 2009, pp. 1 –8.

[7] ——, "Communication-aware navigation functions for cooperative target tracking," in *Proc. IEEE Am. Control Conf.*, June 2009, pp. 1316 –1322.

[8] M. De Gennaro and A. Jadbabaie, "Formation control for a cooperative multi-agent system using decentralized navigation functions," in *Proc. IEEE Am. Control Conf.*, June 2006, p. 6 pp.

[9] D. Dimarogonas and K. Kyriakopoulos, "A feedback stabilization and collision avoidance scheme for multiple independent nonholonomic non-point agents," in *Proc. IEEE Int. Symp. Intell. Control, Mediterr. Conf. Control Autom.*, June 2005, pp. 820 –825.

[10] ——, "Decentralized stabilization and collision avoidance of multiple air vehicles with limited sensing capabilities," in *Proc. IEEE Am. Control Conf.*, June 2005, pp. 4667 – 4672 vol. 7.

[11] J. Chen, W. E. Dixon, M. Dawson, and M. McIntyre, "Homography-based visual servo tracking control of a wheeled mobile robot," *IEEE Trans. Robot.*, vol. 22, no. 2, pp. 406–415, 2006.

[12] D. Dimarogonas, M. Zavlanos, S. Loizou, and K. Kyriakopoulos, "Decentralized motion control of multiple holonomic agents under input constraints," in *Proc. IEEE Conf. Decis. Control*, vol. 4, Dec. 2003, pp. 3390 – 3395.

[13] M. Zavlanos and G. Pappas, "Distributed connectivity control of mobile networks," *IEEE Trans. Robot.*, vol. 24, no. 6, pp. 1416–1428, Dec. 2008.

[14] D. Dimarogonas and K. Johansson, "Decentralized connectivity maintenance in mobile networks with bounded inputs," in *Proc. IEEE Int. Conf. Robot. Autom.*, May 2008, pp. 1507–1512.

[15] Z. Kan, S. Subramanian, J. Shea, and W. E. Dixon, "Vision based connectivity maintenance of a network with switching topology," in *IEEE Multi-conference on Systems and Control, to appear*, 2010.